Math 1552

Section 8.5:

The Method of Partial Fractions

Math 1552 lecture slides adapted from the course materials
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When to Use Partial Fractions:

$$\int \frac{(x^3+2x+1)}{x^2+3}\,dx$$

Use the method of partial fractions to evaluate the integral of a *rational function* when:

- The degree of the numerator is *less than* that of the denominator.
- The denominator can be *completely factored* into linear and/or irreducible quadratic terms – *NO complex numbers in this class!*

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a "1", factor it out.

$$\int \frac{x\,dx}{2x^2+4x+10}$$

→ pull out the factor of two

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a “1”, factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first.

Quick refresher on polynomial long division

Question: What do you when asked to evaluate this integral? $\int \frac{x^3 - 2x^2 - 4}{x-3} dx = I$

Short answer: Observe that $x^3 - 2x^2 - 4 = (x-3)(x^2 + x + 3) + \mathbf{5}$ (How?)

$$
\begin{array}{r|l}
 & x^2 + x + 3 \\
\hline
x - 3 & x^3 - 2x^2 + 0x - 4 \\
 & -(x^3 - 3x^2) \\
\hline
 & x^2 + 0x \\
 & -(x^2 - 3x) \\
\hline
 & 3x - 4 \\
 & -(3x - 9) \\
\hline
 & 5
\end{array}
$$

(This standard method works for denominator polynomials of degree larger than one.)

What this shows is that:

$$x^3-2x^2-4=(x-3)(x^2+x+3)+5$$

$$I=\int(x^2+x+3)\,dx+5\int\frac{dx}{x-3}$$

Partial Fractions Procedure:

1. If the leading coefficient of the denominator is not a “1”, factor it out.
2. If the degree of the numerator is greater than that of the denominator, carry out long division first.
3. Factor the denominator completely into linear and/or irreducible quadratic terms.

Partial Fractions Procedure:

4. For each linear term of the form $(x-a)^k$, you will have *k* partial fractions of the form:

$$\boxed{\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\frac{A_3}{(x-a)^3}+\ldots+\frac{A_k}{(x-a)^k}}$$

(Note: if *k*=1, there is only one fraction to handle, etc.)

$$\int \frac{dx}{(x-1)^k (x-2)^J}$$

$k=1$ and $J=1$

$$\frac{1}{(x-1)(x-2)} = \frac{A}{(x-1)} + \frac{B}{(x-2)}$$

$$\int \frac{dx}{(x-1)^3 (x-2)^2}$$

$k=3$ and $k=2$

$$\frac{1}{(x-1)^3(x-2)^2} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3}$$
$$+ \frac{B_1}{(x-2)} + \frac{B_2}{(x-2)^2}$$

$$\frac{x^2-1}{(x-1)^3(x-2)^2}$$

Partial Fractions Procedure:

5. For each irreducible quadratic term of the form $(x^2+bx+c)^m$, you will have *m* partial fractions of the form:

$$\boxed{\frac{A_1x+B_1}{x^2+bx+c}+\frac{A_2x+B_2}{\left(x^2+bx+c\right)^2}+\frac{A_3x+B_3}{\left(x^2+bx+c\right)^3}+\ldots+\frac{A_mx+B_m}{\left(x^2+bx+c\right)^m}}$$

(Note: if *m*=1, there is only one fraction, etc.)

Ex:
$$\int \frac{dx}{(x-1)^2(x^2+1)^3}$$

$$\frac{1}{(x-1)^2(x^2+1)^3} = \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2}$$

$$+ \frac{C_1x+D_1}{x^2+1} + \frac{C_2x+D_2}{(x^2+1)^2} + \frac{C_3x+D_3}{(x^2+1)^3}$$

Partial Fractions Procedure:

6. Solve for all the constants A_i and B_i. To solve:
   - Multiply everything by the common denominator.
   - Combine all like terms, then solve equations for all the A_i and B_i terms; OR plug in values to find equations for A_i and B_i terms.
7. Integrate using all the integration methods we have learned.

Example 1: Evaluate the integral: $\int \frac{x^3+4x^2}{2x^2+8x-10}\,dx = I$

① factor out the constant

$$I = \frac{1}{2}\int \frac{x^3+4x^2}{x^2+4x-5}\,dx$$

② apply polynomial long div.:

$$\begin{array}{r} x \\ x^2+4x-5\,\overline{\smash{)}\,x^3+4x^2+0x+0} \end{array}$$

$$-\left(x^3+4x^2-5x\right)$$

$$5x+0$$

$$\frac{(x^2+4x-5)x+5x}{x^2+4x-5}=\frac{x^3+4x^2}{x^2+4x-5}$$

3 expand out the integrand:

$$I=\frac{1}{2}\int x\cdot dx+\frac{5}{2}\int\frac{x}{x^2+4x-5}\,dx$$

easy: $\frac{1}{2}\cdot\frac{x^2}{2}+C_1$

apply partial fractions

$$I_2 = -\frac{5}{2}\int\frac{x}{x^2+4x-5}\,dx$$

factor the denominator:

$$x^2+4x-5 = (x-1)(x+5)$$

$$\frac{x}{(x-1)(x+5)} = \frac{A}{x-1} + \frac{B}{x+5}$$

mult. by common denom.

Procedure:

$$\frac{x}{(x-1)(x+5)} * (x-1)(x+5)$$

$$= \left(\frac{A}{x-1} + \frac{B}{x+5}\right) * (x-1)(x+5)$$

Sol: $x = \frac{A\cancel{(x-1)}(x+5)}{\cancel{(x-1)}} + \frac{B(x-1)\cancel{(x+5)}}{\cancel{x+5}}$

$\longleftrightarrow \quad x = A(x+5) + B(x-1) \quad (*)$

- Now we need to plug in specific x values $\longrightarrow$ then solve for A, B

easy values to pick: $x = -5, +1$

with $x=-5$: $-5 = 0 \cdot A - 6B$

$\rightarrow B = 5/6$

with $x=+1$: $1 = 6A + 0 \cdot B$

$\rightarrow A = 1/6$

So since $A = 1/6$, $B = 5/6$, we get the partial fraction decomposition is:

$$\frac{x}{x^2+4x-5} = \frac{1}{6(x-1)} + \frac{5}{6(x+5)}$$

- <u>last step</u>: integrate the result:

$$I_2 = \frac{5}{2}\int \frac{x}{x^2+4x-5}\,dx$$

$$= \frac{5}{12}\int \frac{dx}{x-1} + \frac{25}{12}\int \frac{dx}{x+5}$$

$$= \frac{5}{12}\ln|x-1| + \frac{25}{12}\ln|x+5| + C_2$$

$\rightarrow$ combine to write:

$$I = \frac{x^2}{4} + \frac{5}{12}\ln|x-1| + \frac{25}{12}\ln|x+5| + C$$

Example 2: Evaluate the integral: $\int \frac{x^2-1}{x(x^2+1)^2}\,dx = I$

We want to apply partial fractions:

→ leading term on the denom is one ✓

→ deg(num) = 2, deg(denom) = 3 (No polynomial long division)

→ cannot factor denom.

$x(x^2+1)^2$ any further

$\longrightarrow$ directly apply the partial fractions procedure:

$$\frac{x^2-1}{x(x^2+1)^2} = \frac{A}{x} + \underbrace{\frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}}\ (*)$$

$x = x^k \longleftrightarrow k=1$

$(x^2+1)^J = (x^2+1)^2 \longleftrightarrow J=2$

→ multiply both sides of (*) by the common denominator $x(x^2+1)^2$:

$$x^2-1 = A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x \qquad (**)$$

→ Now what we need to do is to plug in specific values of x to solve for A, B, C, D, E

good values of x to choose:

$x = 0, \pm 1, \pm 2$ into $(**)$

with $x=0$: $-1 = A + 0 + 0 \Leftrightarrow A = -1$

with $x=+1$: $0 = 4A + 2B + 2C + D + E$

$\Leftrightarrow 4 = 2B + 2C + D + E$